GUSHEL–MUKAI MANIFOLDS

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Abstract. A Gushel–Mukai manifold is a smooth Fano n-fold with Picard number 1, index $n-2$, and degree 10. These manifolds were classified by Fano, Gushel, and Mukai a long time ago; in particular, $n \in \{3,4,5,6\}$. We use Mukai's description of these manifolds as (in most cases) complete intersections in the Grassmannian $\mathsf{Gr}(2,V_5)$ to establish in Section 1 their main cohomological properties.

With any Gushel–Mukai manifold, one can associate a 10-dimensional subspace $A \subset \bigwedge^3 V_6$ which is Lagrangian for the conformal symplectic structure given by wedge product. This simple construction establishes a close link between Gushel–Mukai manifolds and Eisenbud–Popescu–Walter sextic hypersurfaces $Y_A \subset \mathbf{P}(V_6)$ associated with A and their canonical double covers $\widetilde{Y}_A \twoheadrightarrow Y_A$ constructed by O'Grady, where $\widetilde{Y}_A$ is a hyper-Kähler fourfold. We explain in Section 2 the results of Debarre–Kuznetsov who used this link to describe the period maps of Gushel–Mukai manifolds.

We examine in Section 3 the rationality question for Gushel–Mukai manifolds. In dimensions 5 and 6, it is classical that they are all rational. In dimensions 3 and 4, the situation is analogous to that of cubic hypersurfaces of the same dimensions: it is known (using the same methods as for cubics) that general Gushel–Mukai threefolds are irrational (and one expects them to all be irrational); some Gushel–Mukai fourfolds are rational but one expects that very general Gushel–Mukai fourfolds are irrational, although no irrational Gushel–Mukai fourfolds are known.

We explain in Section 4 various degenerations of Gushel–Mukai manifolds, the information they provide in the smooth case, and, finally, Iliev's ambitious (and still conjectural) plan to describe the singular locus of the theta divisor of the intermediate Jacobian of any Gushel–Mukai threefold, following the path that Voisin followed for quartic double solids.

In Section 5, we study the derived categories of coherent sheaves on Gushel–Mukai manifolds. We construct semiorthogonal decompositions of these derived categories and study in detail one of the components, called a *Gushel–Mukai category*. We discuss a duality theorem relating the Gushel–Mukai categories of different Gushel–Mukai manifolds.

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1. Gushel–Mukai manifolds

The notation $U_n, V_n, W_n, \ldots$ means a complex vector space of dimension n.

1.1. **Definition.**

Theorem 1.1 (Mukai). *Any smooth Fano n-fold X_n with Picard number 1, index $n-2$, and degree 10 has dimension $n \in \{3,4,5,6\}$ and can be obtained as follows:*

- *either $n \in \{3,4,5\}$ and $X_n = \mathsf{Gr}(2,V_5) \cap \mathbf{P}(W_{n+5}) \cap Q \subset \mathbf{P}(\bigwedge^2 V_5)$, where Q is a quadric and $W_{n+5} \subset \bigwedge^2 V_5$ is a vector subspace of dimension $n+5$ (*ordinary *case);*
- *or $X_n \longrightarrow \mathsf{Gr}(2,V_5) \cap \mathbf{P}(W_{n+4})$ is a double cover branched along an X_{n-1} (*special *case).*

The following definition covers both cases.

Definition 1.2. A Gushel–Mukai manifold of dimension n (for $n \in \{3,4,5,6\}$) is a smooth dimensionally transverse intersection

$$X_n = \mathrm{Cone}(\mathsf{Gr}(2,V_5)) \cap \mathbf{P}(W_{n+5}) \cap Q \subset \mathbf{P}(\bigwedge^2 V_5 \oplus \mathbf{C}),$$

where Q is a quadric hypersurface and $W_{n+5} \subset \mathbf{C} \oplus \bigwedge^2 V_5$ is a linear subspace of dimension $n+5$.[1]

The projection $\gamma\colon X_n \longrightarrow \mathsf{Gr}(2,V_5) \subset \mathbf{P}(\bigwedge^2 V_5)$ from the vertex of the cone is called the *Gushel morphism* and X_n is special or ordinary depending on whether the vertex lies in $\mathbf{P}(W_{n+5})$ or not. In both cases, $\gamma^*\mathscr{O}_{\mathsf{Gr}(2,V_5)}(1)$ is a generator of $\mathrm{Pic}(X_n)$.

1.2. **Moduli spaces.** We stick to the ordinary case for simplicity. In that case, we may consider W_{n+5} as a subspace of $\bigwedge^2 V_5$ and an ordinary Gushel–Mukai manifold of dimension n is a smooth complete intersection $X = \mathsf{Gr}(2,V_5) \cap \mathbf{P}(W_{n+5}) \cap Q$. The intersection $M := \mathsf{Gr}(2,V_5) \cap \mathbf{P}((W_{n+5})$ is also smooth.

Let $W := W_{n+5}$. There is an isomorphism

$$\begin{array}{rccc} q\colon & V_5 & \xrightarrow{\sim} & H^0(\mathbf{P}(W), \mathscr{I}_M(2)) \\ & v & \longmapsto & \big(w \longmapsto v \wedge w \wedge w\big) \end{array}$$

so that V_5 can be identified with the set of Plücker quadrics (those containing M) in W. We set

$$V_6 := H^0(\mathbf{P}(W), \mathscr{I}_X(2)) \simeq V_5 \oplus \mathbf{C}Q. \tag{1.1}$$

Let $Q(v) \subset \mathbf{P}(W)$ be the quadric defined by $q(v)$, for any nonzero $v \in V_6$. This way, we recover

$$M = \bigcap_{v \in V_5} Q(v) \quad \text{and} \quad X = \bigcap_{v \in V_6} Q(v)$$

from the *Gushel–Mukai data set* (V_6, V_5, W, q).

Pick $v_0 \in V_6 \smallsetminus V_5$ and consider the kernel A of the map

$$\begin{array}{ccc} \bigwedge^3 V_5 \oplus W & \longrightarrow & W^\vee \\ (\xi, w) & \longmapsto & \big(w' \longmapsto \xi \wedge w' + q(v_0)(w,w')\big). \end{array} \tag{1.2}$$

We use the decomposition $\bigwedge^3(V_5 \oplus \mathbf{C}v_0) \simeq \bigwedge^3 V_5 \oplus (\bigwedge^2 V_5 \otimes v_0)$ to think of A as a subspace of $\bigwedge^3 V_6$. It is straightforward to check that A is Lagrangian with respect to the conformal symplectic structure on $\bigwedge^3 V_6$ given by exterior product.

We call such a triple $(V_6, V_5 \subset V_6, A \subset \bigwedge^3 V_6$ Lagrangian$)$ a *Lagrangian data set.*

Proposition 1.3. *The subspace A contains no decomposable vectors, that is, no nonzero vectors of the form $v_0 \wedge v_1 \wedge v_2$ with $v_0, v_1, v_2 \in V_6$.*

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[1] In Section 5, we will allow $n = 2$, in which case X is a (smooth) K3 surface of degree 10.

Instead of giving a complete proof (see [DK1] for more general statements and proofs), we just highlight the link between decomposable vectors of A and singularities of X. Let $v_0 \wedge v_1 \wedge v_2 \in A$ be a decomposable vector. We can assume $v_1, v_2 \in V_5$; moreover, one can prove that v_0 is not in V_5 by studying the ranks of forms in $W^\perp$. Hence $v_1 \wedge v_2 \in \ker(q(v_0))$, so $Q(v_0)$ is singular at $[v_1 \wedge v_2] \in \mathbf{P}(W)$ hence so is X. One can prove the converse: any singularity of X comes from the singularity of some quadric $Q(v_0)$ at a point $[v_1 \wedge v_2]$.

The surprising fact that makes this construction useful is that the process is reversible: for *any* Lagrangian data set (V_6, V_5, A) such that A contains no decomposable vectors (this holds for A general), one can define a (smooth) ordinary Gushel–Mukai manifold. In [DK3], the authors combine the previous constructions and prove the following.

Theorem 1.4 (Debarre–Kuznetsov)**.** *There exists a coarse moduli space (in fact, a smooth Deligne–Mumford stack) $\mathbf{M}_n^{\mathrm{GM}}$ for Gushel–Mukai manifolds of dimension $n \in \{3,4,5,6\}$, which is quasi-projective and irreducible of dimension $25-(5-n)(6-n)/2$. Moreover, ordinary (resp. special) Gushel–Mukai manifolds are parametrized by an open (resp. closed) subspace $\mathbf{M}_n^{\mathrm{GM,ord}}$ (resp. $\mathbf{M}_n^{\mathrm{GM,spe}}$) of $\mathbf{M}_n^{\mathrm{GM}}$.*

1.3. **Hodge diamonds.** With some work, the Hodge numbers of Gushel–Mukai manifolds can be computed (see [DK1, Propositions 3.1 and 3.4]).

Proposition 1.5. *The integral cohomology of a Gushel–Mukai manifold of dimension n is torsion-free and its Hodge diamond is*

$$(n=3)\qquad \begin{array}{c} 1\\ 0\quad 0\\ 0\quad 1\quad 0\\ 0\quad 10\quad 10\quad 0\\ 0\quad 1\quad 0\\ 0\quad 0\\ 1 \end{array}$$

$$(n=4)\qquad \begin{array}{c} 1\\ 0\quad 0\\ 0\quad 1\quad 0\\ 0\quad 0\quad 0\quad 0\\ 0\quad 1\quad 22\quad 1\quad 0\\ 0\quad 0\quad 0\quad 0\\ 0\quad 1\quad 0\\ 0\quad 0\\ 1 \end{array}$$

$$(n=5)\qquad \begin{array}{c} 1\\ 0\quad 0\\ 0\quad 1\quad 0\\ 0\quad 0\quad 0\quad 0\\ 0\quad 0\quad 2\quad 0\quad 0\\ 0\quad 0\quad 10\quad 10\quad 0\quad 0\\ 0\quad 0\quad 2\quad 0\quad 0\\ 0\quad 0\quad 0\quad 0\\ 0\quad 1\quad 0\\ 0\quad 0\\ 1 \end{array}$$

$$(n=6)\qquad \begin{array}{c} 1\\ 0\quad 0\\ 0\quad 1\quad 0\\ 0\quad 0\quad 0\quad 0\\ 0\quad 0\quad 2\quad 0\quad 0\\ 0\quad 0\quad 0\quad 0\quad 0\quad 0\\ 0\quad 0\quad 1\quad 22\quad 1\quad 0\quad 0\\ 0\quad 0\quad 0\quad 0\quad 0\quad 0\\ 0\quad 0\quad 2\quad 0\quad 0\\ 0\quad 0\quad 0\quad 0\\ 0\quad 1\quad 0\\ 0\quad 0\\ 1 \end{array}$$

In particular, only the middle cohomology $H^n(X, \mathbf{Z})$ is interesting (in other degrees, it is induced from the cohomology of $\mathsf{Gr}(2, V_5)$ by the Gushel map $\gamma\colon X \longrightarrow \mathsf{Gr}(2, V_5)$). We define the *vanishing cohomology* by

$$H^n(X, \mathbf{Z})_{\mathrm{van}} \coloneqq \left(\gamma^* H^n(\mathsf{Gr}(2, V_5), \mathbf{Z})\right)^\perp \subset H^n(X, \mathbf{Z}).$$

The Hodge numbers for the vanishing cohomology are therefore

$(n=3)$	$(n=4)$	$(n=5)$	$(n=6)$
0 10 10 0	0 1 20 1 0	0 0 10 10 0 0	0 0 1 20 1 0 0

1.4. **Period maps.** Let X be a Gushel–Mukai manifold of dimension n. When $n \in \{3,5\}$, the Hodge structure $H^n(X)$ has level 1. There is a 10-dimensional principally polarized intermediate Jacobian

$$\mathrm{Jac}(X) := H^n(X,\mathbf{C})/\big(H^{(n+1)/2,(n-1)/2}(X) + H^n(X,\mathbf{Z})\big)$$

and a period map

$$\begin{array}{rcl} \wp_n \colon \mathbf{M}_n^{\mathrm{GM}} & \longrightarrow & \mathscr{A}_{10} \\ {[X]} & \longmapsto & [\mathrm{Jac}(X)]. \end{array}$$

Assume now $n \in \{4,6\}$. If $n=4$, the vanishing cohomology lattice $(H^n(X,\mathbf{Z})_{\mathrm{van}}, \smile)$ is isomorphic to the even lattice

$$\Lambda := E_8^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 2} \oplus \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

where E_8 is the rank-8 positive definite even lattice. It has signature $(20,2)$. If $n=6$, the vanishing cohomology lattice is isomorphic to $\Lambda(-1)$ (the same lattice with the opposite intersection form).

The manifold

$$\Omega := \{\omega \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid (\omega \cdot \omega) = 0 \ , (\omega \cdot \bar{\omega}) < 0\}$$

is a homogeneous space with two components, Ω^+ and Ω^-, both isomorphic to the 20-dimensional open complex manifold $\mathrm{SO}_0(20,2)/\mathrm{SO}(20) \times \mathrm{SO}(2)$, a bounded symmetric domain of type IV. The quotient

$$\mathscr{D} := \widetilde{O}(\Lambda)\backslash\Omega^+, \tag{1.3}$$

where $\widetilde{O}(\Lambda)$ is a subgroup of index 2, called the *stable orthogonal group,* of the isometry group $O(\Lambda)$ of the lattice Λ, has the structure of an irreducible quasi-projective variety of dimension 20. The domain $\mathscr{D}$ carries a nontrivial canonical involution $r_{\mathscr{D}}$ associated with the double cover $\mathscr{D} \twoheadrightarrow O(\Lambda)\backslash\Omega^+$.

A marking of X is an isometry $\psi \colon H^n(X,\mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} \Lambda((-1)^{\frac{n}{2}})$. The Hodge structure $H^n(X)_{\mathrm{van}}$ is of K3 type and there is a period map

$$\begin{array}{rcl} \wp_n \colon \mathbf{M}_n^{\mathrm{GM}} & \longrightarrow & \mathscr{D} \\ {[X]} & \longmapsto & [\psi_{\mathbf{C}}(H^{n/2+1,n/2-1}(X))], \end{array}$$

where ψ is any marking on X.

Altogether, there are period maps

$$
\begin{tikzcd}
\text{dim.} & & & \text{dim.} \\
22 & \mathbf{M}_3^{\mathrm{GM}} \arrow[dr, "\wp_3"] & & \\
24 & \mathbf{M}_4^{\mathrm{GM}} \arrow[dr, "\wp_4"] & \mathscr{A}_{10} & 55 \\
25 & \mathbf{M}_5^{\mathrm{GM}} \arrow[ur, "\wp_5"'] & \mathscr{D} & 20 \\
25 & \mathbf{M}_6^{\mathrm{GM}} \arrow[ur, "\wp_6"'] & &
\end{tikzcd}
$$

They are dominant when n is even. The kernel of the differentials of $\wp_n$ have the following dimensions: 5 when $n \in \{5, 6\}$, 4 when $n = 4$, and 2 when $n = 3$. This is also the dimension of the fibers of $\wp_n$.

The construction of special Gushel–Mukai manifolds produces maps $\mathbf{M}_n^{\mathrm{GM,ord}} \longrightarrow \mathbf{M}_{n+1}^{\mathrm{GM,spe}} \subset \mathbf{M}_{n+1}^{\mathrm{GM}}$ for $n \in \{3, 4, 5\}$, which we see as rational maps $q_n \colon \mathbf{M}_n^{\mathrm{GM}} \dashrightarrow \mathbf{M}_{n+1}^{\mathrm{GM}}$. We will construct in Sections 2.2, 2.3, and 2.4 maps $p_n \colon \mathbf{M}_n^{\mathrm{GM}} \longrightarrow \mathscr{D}$ and a rational map $a \colon \mathscr{D} \dashrightarrow \mathscr{A}_{10}$ (see (2.4)) that fit into a diagram

$$
\begin{tikzcd}
\mathbf{M}_3^{\mathrm{GM}} \arrow[d, dashed, "q_3"'] \arrow[ddr, "p_3"] & & \\
\mathbf{M}_4^{\mathrm{GM}} \arrow[dd, dashed, "q_4"'] \arrow[dr, "p_4"] & & \\
& \mathscr{D} \arrow[r, dashed, "a"] & \mathscr{A}_{10} \\
\mathbf{M}_5^{\mathrm{GM}} \arrow[d, dashed, "q_5"'] \arrow[ur, "p_5"'] & & \\
\mathbf{M}_6^{\mathrm{GM}} \arrow[uur, "p_6"'] & &
\end{tikzcd}
$$

with commutative triangles[2] and such that $\wp_4 = p_4$, $\wp_6 = p_6$, and $\wp_3 = a \circ p_3$, $\wp_5 = a \circ p_5$.

2. Lagrangians and EPW sextics

2.1. **EPW sextics.** Let $A \subset \bigwedge^3 V_6$ be a (10-dimensional) subspace which is Lagrangian for the conformal symplectic structure on $\bigwedge^3 V_6$ given by wedge product.

Definition 2.1. For any integer ℓ, we set

$$Y_A^{\geqslant \ell} := \big\{[U_1] \in \mathbf{P}(V_6) \mid \dim\big(A \cap (U_1 \wedge \textstyle\bigwedge^2 V_6)\big) \geqslant \ell\big\} \tag{2.1}$$

and

$$Y_A^{\ell} := Y_A^{\geqslant \ell} \smallsetminus Y_A^{\geqslant \ell+1}.$$

[2] The vertical rational maps q_n on the left cannot be composed: they are defined on the open subsets of ordinary Gushel–Mukai manifolds but their images consist of special Gushel–Mukai manifolds.

Theorem 2.2 (O'Grady)**.** *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace. If A contains no decomposable vectors,*

(a) *Y_A is an integral normal sextic hypersurface in $\mathbf{P}(V_6)$ called an* EPW sextic*;*
(b) *$Y_A^{\geqslant 2} = \mathrm{Sing}(Y_A)$ is an integral normal Cohen–Macaulay surface of degree* 40;
(c) *$Y_A^{\geqslant 3} = \mathrm{Sing}(Y_A^{\geqslant 2})$ is finite, and is empty for A general;*
(d) *$Y_A^{\geqslant 4}$ is empty.*

We may also consider the orthogonal $A^\perp \subset \bigwedge^3 V_6^\vee$. It is still Lagrangian and

$$Y_{A^\perp}^{\geqslant \ell} = \{[U_5] \in \mathbf{P}(V_6^\vee) \mid \dim(A \cap \textstyle\bigwedge^3 U_5) \geqslant \ell\}. \tag{2.2}$$

The properties of EPW sextics that are of most interest come from the existence of a (finite) canonical double cover $f_A \colon \widetilde{Y}_A \longrightarrow Y_A$ ([O2, Section 1.2]) with the following properties.

Theorem 2.3 (O'Grady)**.** *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace which contains no decomposable vectors and let $Y_A \subset \mathbf{P}(V_6)$ be the associated EPW sextic.*

(a) *The double cover $f_A \colon \widetilde{Y}_A \longrightarrow Y_A$ is branched over the surface $Y_A^{\geqslant 2}$ and induces the universal cover of Y_A^1.*
(b) *The variety $\widetilde{Y}_A$ is irreducible and normal, and its singular locus is the finite set $f_A^{-1}(Y_A^{\geqslant 3})$.*
(c) *When $Y_A^{\geqslant 3}$ is empty, $\widetilde{Y}_A$ is a smooth hyper-Kähler fourfold which is a deformation of the Hilbert square of a K3 surface.*

Proof. Item (a) was proved in [O2, Proof of Theorem 4.15, p. 179], item (b) follows from statement (3) in the introduction of [O2], and item (c) is [O1, Theorem 1.1(2)]. □

In [DK2], the authors generalized the construction of the double cover of Theorem 2.3 and they defined another canonical double cover of the surface $Y_A^{\geqslant 2}$ which will be used in Section 2.4.

Theorem 2.4 (Debarre–Kuznetsov)**.** *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace which contains no decomposable vectors and let $Y_A \subset \mathbf{P}(V_6)$ be the associated EPW sextic. There exists a double cover*

$$\pi_A \colon \widetilde{Y}_A^{\geqslant 2} \longrightarrow Y_A^{\geqslant 2}$$

branched over $Y_A^{\geqslant 3}$, where the surface $\widetilde{Y}_A^{\geqslant 2}$ is integral, normal, and smooth away from $\pi_A^{-1}(Y_A^{\geqslant 3})$.

Proof. These constructions are explained in [DK2, Sections 3 and 4]. They are a bit technical but the idea is that locally, isotropic degeneracy loci (such as $Y_A^{\geqslant k}$, the locus where two families of isotropic subspaces meet in dimension $\geqslant k$) are quadratic degeneracy loci (loci where a family of quadratic forms has rank $\leqslant k$). When the rank is even, the latter have canonical double coverings parametrizing connected components of the family of linear subspaces of maximal dimension; these coverings are branched along the locus where the rank drops by 1 more. □

Finally, O'Grady constructed a GIT moduli space $\mathbf{M}^{\mathrm{EPW}}$ for Lagrangians $A \subset \bigwedge^3 V_6$ with no decomposable vectors and there is an (extended) period map

$$\begin{aligned} \wp\colon \mathbf{M}^{\mathrm{EPW}} &\longrightarrow \mathscr{D} \\ [A] &\longmapsto [\widetilde{Y}_A], \end{aligned}$$

where $\mathscr{D}$ is the period domain already defined in (1.3) (one has to be a bit careful because $\widetilde{Y}_A$ acquires singularities when $Y_A^{\geqslant 3}$ becomes nonempty). This map is an open embedding (by the Verbitsky–Markman Torelli theorem, at least when $\widetilde{Y}_A$ is smooth).

2.2. **Gushel–Mukai manifolds and Lagrangians.** Recall from Section 1.2 that we can associate with any ordinary Gushel–Mukai n-fold $X \subset \mathbf{P}(\bigwedge^2 V_5)$ a Lagrangian subspace $A \subset \bigwedge^3 V_6$, where $V_6 := V_5 \oplus \mathbf{C}Q$ is the set of quadrics that contain X and the hyperplane V_5 is in the stratum $Y^{5-n}_{A^\perp}$ defined in (2.2).

Corollary 2.5. *For each $n \in \{3,4,5\}$, there is a surjective morphism*

$$p_n\colon \mathbf{M}_n^{\mathrm{GM,ord}} \longrightarrow \mathbf{M}^{\mathrm{EPW}}$$

between moduli spaces. The fiber $p_n^{-1}(A)$ is isomorphic to $Y^{5-n}_{A^\perp}$ (modulo automorphisms).

We stick here to the case of ordinary Gushel–Mukai manifolds, but it is easy to include special Gushel–Mukai manifolds in that statement (just "glue" p_n and p_{n-1}).

When $n = 6$, one has $\mathbf{M}_6^{\mathrm{GM}} = \mathbf{M}_5^{\mathrm{GM,ord}}$, so we can define p_6 as p_5.

2.3. **Comparison of the period maps (even-dimensional case).**

Theorem 2.6 (Debarre–Kuznetsov). *Let $n \in \{4,6\}$. The period map for Gushel–Mukai n-folds factors as*

$$\wp_n\colon \mathbf{M}_n^{\mathrm{GM}} \xrightarrow{p_n} \mathbf{M}^{\mathrm{EPW}} \hookrightarrow \mathscr{D}.$$

Sketch of proof. Given a Gushel–Mukai fourfold X, one considers the (3-dimensional) family $F_1(X)$ of lines contained in X and the incidence correspondence

$$\begin{tikzcd} & I(X) = \{(x,\ell) \in X \times F_1(X) \mid x \in \ell\} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & F_1(X). \end{tikzcd}$$

Under (explicit) generality assumptions on the Lagrangian A associated with X and the Plücker point $p_X := [V_5] \in \mathbf{P}(V_6^\vee)$, one proves that $F_1(X)$ is smooth, so that there is an induced map

$$p_*q^*\colon H^4(X,\mathbf{Z}) \longrightarrow H^2(F_1(X),\mathbf{Z}).$$

The construction of a relation between $F_1(X)$ and $\widetilde{Y}_A$ goes as follows. For any line $\ell \subset X$, there is a unique $[v_\ell] \in \mathbf{P}(V_5)$ such that ℓ lies in the maximal linear space $\mathbf{P}(v_\ell \wedge V_5)$ of $\mathsf{Gr}(2, V_5)$, hence there is a natural map

$$\begin{array}{rcl} F_1(X) & \longrightarrow & \mathbf{P}(V_5) \\ \ell & \longmapsto & [v_\ell] \end{array}$$

whose image is contained in $Y_A \cap \mathbf{P}(V_5)$. This map factors through the canonical double cover $\widetilde{Y}_{A,V_5} := f_A^{-1}(Y_A \cap \mathbf{P}(V_5)) \longrightarrow Y_A \cap \mathbf{P}(V_5)$ and the resulting map $F_1(X) \longrightarrow \widetilde{Y}_{A,V_5}$ is a small resolution of the threefold $\widetilde{Y}_{A,V_5}$.

So there are maps

$$H^2(\widetilde{Y}_A, \mathbf{Z}) \xrightarrow{\sim} H^2(\widetilde{Y}_{A,V_5}, \mathbf{Z}) \hookrightarrow H^2(F_1(X), \mathbf{Z}). \tag{2.3}$$

One shows that the image $p_*q^*(H^4(X, \mathbf{Z})_{\mathrm{van}})$ is contained in the image of the map (2.3) and that there is an induced morphism

$$H^4(X, \mathbf{Z})_{\mathrm{van}} \longrightarrow H^2(\widetilde{Y}_A, \mathbf{Z})_{\mathrm{prim}}.$$

One next shows that this map is a Hodge isometry for the cup-product on the left and the form $-q_{\widetilde{Y}_A}$ on the right: this proves the statement for a general X. But then the result holds for any smooth X since it holds over a nonempty open subset of $\mathbf{M}_n^{\mathrm{GM}}$, which is irreducible.

A similar method is used for Gushel–Mukai sixfolds X, using the scheme $F_2^\sigma(X)$ of so-called σ-planes (of the type $\mathbf{P}(V_1 \wedge V_4)$) contained in X. Under the same generality assumptions, one shows that $F_2^\sigma(X)$ is a $\mathbf{P}^1$-fibration over the threefold $\widetilde{Y}_{A,V_5} \subset \widetilde{Y}_A$ and that both are smooth. As in the fourfold case, there is an Abel–Jacobi map (the map p is a $\mathbf{P}^2$-fibration)

$$p_*q^* \colon H^6(X, \mathbf{Z}) \longrightarrow H^2(F_2^\sigma(X), \mathbf{Z}),$$

maps

$$H^2(\widetilde{Y}_A, \mathbf{Z}) \xrightarrow{\sim} H^2(\widetilde{Y}_{A,V_5}, \mathbf{Z}) \hookrightarrow H^2(F_2^\sigma(X), \mathbf{Z}),$$

and an induced morphism

$$H^6(X, \mathbf{Z})_{\mathrm{van}} \longrightarrow H^2(\widetilde{Y}_A, \mathbf{Z})_{\mathrm{prim}}.$$

Again, one proves that this map is a Hodge isometry for the cup-product on the left and the form $q_{\widetilde{Y}_A}$ on the right. □

2.4. **Comparison of the period maps (odd-dimensional case).** In the odd-dimensional case, one needs to relate the 10-dimensional intermediate Jacobian to some abelian variety attached to A. The surface $Y_A^{\geqslant 2}$ is regular, but the surface $\widetilde{Y}_A^{\geqslant 2}$ has irregularity 10 and its Albanese variety will play a crucial role.

Theorem 2.7 (Debarre–Kuznetsov)**.** *Let X be a Gushel–Mukai manifold of dimension $n \in \{3,5\}$ with associated Lagrangian A. Assume $Y_A^{\geqslant 3} = \varnothing$. There is a canonical isomorphism*

$$\mathrm{Jac}(X) \simeq \mathrm{Alb}(\widetilde{Y}_A^{\geqslant 2}).$$

This isomorphism defines a canonical polarization on the Albanese variety $\mathrm{Alb}(\widetilde{Y}_A^{\geqslant 2})$ which is independent of X, hence a morphism[3]

$$\mathrm{alb}\colon \mathbf{M}^{\mathrm{EPW}} \longrightarrow \mathscr{A}_{10}$$

which factors through the duality involution. The period map for Gushel–Mukai n-folds factors as

$$\wp_n\colon \mathbf{M}_n^{\mathrm{GM}} \xrightarrow{p_n} \mathbf{M}^{\mathrm{EPW}} \xrightarrow{\mathrm{alb}} \mathscr{A}_{10}.$$

We explained at the end of Section 2.1 that the period map for double EPW sextics makes $\mathbf{M}^{\mathrm{EPW}}$ into a dense open subset of $\mathscr{D}$. We may therefore view the map alb as a rational map

$$a\colon \mathscr{D} \dashrightarrow \mathscr{A}_{10}. \tag{2.4}$$

Since we know (by the computation of its differential) that the fibers of the map $\wp_3$ all have dimension 2, the map alb has finite fibers and it has even degree; we expect this degree to be exactly 2 and the factorization

$$\overline{\mathrm{alb}}\colon \mathbf{M}^{\mathrm{EPW}} / \text{ duality involution } \longrightarrow \mathscr{A}_{10} \tag{2.5}$$

to be injective (or the induced rational map $\overline{a}\colon \mathscr{D}/r_{\mathscr{D}} \dashrightarrow \mathscr{A}_{10}$ to be generically injective).

The proofs of Debarre–Kuznetsov are a bit complicated. A simpler (and more natural) way of proving the theorem when X has dimension 3 would be to use the family $C_2^0(X)$ of conics contained in X. It was proved by Logachev in the 80s that it is isomorphic to the blow up of the smooth surface $\widetilde{Y}_A^{\geqslant 2}$ at one point. This defines an Abel–Jacobi map $\mathrm{Alb}(\widetilde{Y}_A^{\geqslant 2}) \longrightarrow \mathrm{Jac}(X)$ and one needs to check that it is an isomorphism (this is perhaps not as simple as it looks since one needs to understand the cohomology of the total space of the family of conics, and this is not anymore a $\mathbf{P}^1$-bundle on the base!).

All the proofs above rely on the construction of appropriate Abel–Jacobi maps. Debarre–Kuznetsov also give an alternative proof of the result for fivefolds. The incidence correspondence between the Hilbert scheme of σ-planes $F_2^\sigma(X)$ and X, given by the universal family of σ-planes in X, induces a map $H_1(F_2^\sigma(X), \mathbf{Z}) \longrightarrow H_5(X, \mathbf{Z})$. For X general, the curve $F_2^\sigma(X)$ is smooth connected (of genus 161) and isomorphic to the hyperplane section $\widetilde{Y}_{A,V_5}^{\geqslant 2}$. Moreover, the map is surjective, so the induced map between abelian varieties

$$\varphi\colon \mathrm{Jac}(\widetilde{Y}_{A,V_5}^{\geqslant 2}) \longrightarrow \mathrm{Jac}(X)$$

is surjective, with connected kernel.

[3]This map, a priori defined only on the locus of Lagrangians A such that $Y_A^{\geqslant 3} = \varnothing$, actually extends.

Theorem 2.8 (Debarre–Kuznetsov, Simplicity argument)**.** *The morphism φ above factors as*

$$\mathrm{Jac}(\widetilde{Y}^{\geqslant 2}_{A,V_5}) \twoheadrightarrow \mathrm{Alb}(\widetilde{Y}^{\geqslant 2}_{A}) \xrightarrow{\sim} \mathrm{Jac}(X).$$

Sketch of proof. We choose X such that V_5 is a very general hyperplane inside V_6; in particular, the curve $\widetilde{Y}^{\geqslant 2}_{A,V_5}$ is smooth. The morphism $\mathrm{Jac}(\widetilde{Y}^{\geqslant 2}_{A,V_5}) \to \mathrm{Alb}(\widetilde{Y}^{\geqslant 2}_{A})$ is surjective by the Lefschetz theorem; we call K its (connected) kernel.

Using the very generality assumption, one can prove that K has two simple factors, of respective dimensions 81 and 70. Since $\mathrm{Jac}(X)$ has dimension 10, this means that $\varphi(K) = 0$; the kernel of φ has the same dimension as K and is connected, thus it is equal to K. By continuity, the result holds for any X. □

This argument cannot be used for threefolds, since in that case the corresponding hyperplane V_5 is never very general inside V_6.

3. Rationality of Gushel–Mukai manifolds

3.1. **Dimensions 5 or 6.** All Gushel–Mukai manifolds of dimensions 5 or 6 are rational (classical).

3.2. **Dimension 4.** The situation in dimension 4 is very similar to that of cubic fourfolds: some rational examples are known (see below), one expects a very general Gushel–Mukai fourfold to be irrational, but no irrational examples are known. Since all Gushel–Mukai fourfolds in the same fiber of the map

$$p_4 \colon \mathbf{M}^{\mathrm{GM}}_4 \longrightarrow \mathbf{M}^{\mathrm{EPW}}$$

are birationally isomorphic, rationality only depends on the associated Lagragian, or on the period point in $\mathscr{D}$.

Example 3.1. Gushel–Mukai fourfolds X containing a σ-plane P (a plane in $\mathsf{Gr}(2, V_5)$ of the form $\mathbf{P}(V_1 \wedge V_4)$) were already studied by Roth (1949) and Prokhorov (1993). They form a codimension-2 family in the moduli space $\mathbf{M}^{\mathrm{GM}}_4$ that dominates via the period map a Heegner divisor in the period domain $\mathscr{D}$. These Gushel–Mukai fourfolds are all rational.

This can be seen as follows (for X general): let $\widetilde{X} \to X$ be the blow-up of P; the projection from P induces a birational morphism $\widetilde{X} \to Y$, where $Y \subset \mathbf{P}^5$ is a smooth quadric, which is the blow-up of a smooth degree-9 surface $\widetilde{S} \subset Y$, itself the blow-up of a smooth degree-10 K3 surface $S \subset \mathbf{P}^6$ at one point. Conversely, starting from a general degree-10 K3 surface $S \subset \mathbf{P}^6$, a general point p on S, and a smooth quadric Y containing the projection $\widetilde{S} \subset \mathbf{P}^5$ from p, the linear system of cubics containing $\widetilde{S}$ gives a birational isomorphism $Y \dashrightarrow X$, where X is a Gushel–Mukai fourfold.

The K3 surface S is associated with X in the sense of Hassett: the Hodge structure $H^2(S)_{\mathrm{prim}}$ sits in the Hodge structure $H^4(X)_{\mathrm{van}}$ (hence also in the Hodge structure $H^2(\widetilde{Y}_A)_{\mathrm{prim}}$).

Example 3.2. Gushel–Mukai fourfolds X containing a ρ-plane P (a plane in $\mathsf{Gr}(2, V_5)$ of the form $\mathsf{Gr}(2, V_3)$) were also studied by Roth. They form a codimension-3 family in the moduli space $\mathbf{M}_4^{\mathrm{GM}}$ that dominates via the period map a Heegner divisor in the period domain $\mathscr{D}$. A general X with this property is birationally isomorphic to a cubic fourfold containing a smooth cubic surface scroll (those were studied by Hassett–Tschinkel and are not expected to be rational).

This can be seen as follows (for X general): the projection from P induces a birational map $X \overset{\sim}{\dashrightarrow} Y$, where $Y \subset \mathbf{P}^5$ is a smooth cubic fourfold. Conversely, a general cubic fourfold $Y \subset \mathbf{P}^5$ containing a smooth cubic scroll contains two families (each parametrized by $\mathbf{P}^2$) of such surfaces. For each such smooth cubic scroll, one can reverse the construction above and produce a smooth Gushel–Mukai fourfold X containing a ρ-plane.

3.3. **Dimension 3.** A general Gushel–Mukai threefold X is irrational (it is expected that they all are). By the Clemens–Griffiths criterion, it is enough to show that $\mathrm{Jac}(X)$ is not the Jacobian of a curve for *one* X (or one degeneration thereof). There are several ways to do that:

- When the Gushel–Mukai threefold X acquires a node, $\mathrm{Jac}(X)$ was shown by Beauville in his thesis to become a $\mathbf{C}^\star$-extension of a 9-dimensional Prym variety associated with a double étale cover of a plane sextic curve (see Section 4.1). Using Mumford's description of the singularities of the theta divisor of a Prym variety, one sees easily that this Prym variety is not the Jacobian of a curve. This implies that the intermediate Jacobian of a general (smooth) Gushel–Mukai threefold X is not the Jacobian of a curve, hence X is irrational (we explain this in more details in Section 4.1).
- Using another degeneration (where X remains smooth; see Section 4.2) and results of Welters and Voisin, one gets another proof of the irrationality of a general Gushel–Mukai threefold (one shows that the singular locus of the theta divisor of the intermediate Jacobian of a general Gushel–Mukai threefold has dimension $\leqslant 5$; see Section 4.2 for more details).
- Debarre and Mongardi found a (smooth) Gushel–Mukai threefold X with a faithful $\mathrm{PSL}(2, \mathbf{F}_{11})$-action. Then $\mathrm{Jac}(X)$ also has a faithful $\mathrm{PSL}(2, \mathbf{F}_{11})$-action and this is too many automorphisms for it to be the Jacobian of a curve. This gives an explicit example of an irrational Gushel–Mukai threefold.

We will see a bit more about the singular locus of the theta divisor of the 10-dimensional intermediate Jacobian of a Gushel–Mukai threefold in the next section.

4. Degenerations

Degenerations may be used to relate Gushel–Mukai manifolds (mostly of dimension 3) and double EPW sextics to situations studied in the past.

4.1. **The nodal degeneration.** Recall that the smoothness of a Gushel–Mukai manifold is equivalent to its associated Lagrangian having no decomposable vectors. Accordingly, when a Lagrangian A is general with a (single) decomposable vector $\bigwedge^3 B$, the associated Gushel–Mukai manifolds (of all dimensions) corresponding to a hyperplane $V_5 \subset V_6$ not containing B acquire an ordinary double point at the point $[V_5 \cap B] \in \mathsf{Gr}(2, V_5)$.

More precisely, given such an A, one can construct a Verra threefold $T_A \subset \mathbf{P}(B) \times \mathbf{P}(B^\perp)$ (a divisor of type $(2,2)$) such that

- the discriminant of the conic bundle $T_A \longrightarrow \mathbf{P}(B)$ is the sextic curve $C_A := Y_A^{\geqslant 2} \cap \mathbf{P}(B)$;
- the discriminant of the conic bundle $T_A \longrightarrow \mathbf{P}(B^\perp)$ is the sextic curve $C_{A^\perp} := Y_{A^\perp}^{\geqslant 2} \cap \mathbf{P}(B^\perp)$.

If $Y_A^{\geqslant 3}$ and $Y_{A^\perp}^{\geqslant 3}$ are both empty, which we assume, these curves are smooth and T_A is smooth. Moreover,

- when $[V_5] \in Y_{A^\perp}^2 \smallsetminus \mathbf{P}(B^\perp)$, the corresponding Gushel–Mukai threefold is birationally isomorphic to T_A;
- when $[V_5] \in Y_{A^\perp}^1 \smallsetminus \mathbf{P}(B^\perp)$, the corresponding Gushel–Mukai fourfold is birationally isomorphic to the double cover of $\mathbf{P}(B) \times \mathbf{P}(B^\perp)$ branched along T_A (a Verra fourfold).

The irrationality of Verra fourfolds is still an open question. As to Verra threefolds, their (9-dimensional) intermediate Jacobians were much studied by Verra through the fact that they are Prym varieties: if $\widetilde{C}_A \longrightarrow C_A$ and $\widetilde{C}_{A^\perp} \longrightarrow C_{A^\perp}$ are the discriminant étale double covers, one has

$$\operatorname{Jac}(T_A) \simeq \operatorname{Prym}(\widetilde{C}_A / C_A) \simeq \operatorname{Prym}(\widetilde{C}_{A^\perp} / C_{A^\perp}).$$

It is expected that these double étale covers are induced by the canonical double étale covers $\pi_A \colon \widetilde{Y}_A^{\geqslant 2} \longrightarrow Y_A^{\geqslant 2}$ and $\pi_{A^\perp} \colon \widetilde{Y}_{A^\perp}^{\geqslant 2} \longrightarrow Y_{A^\perp}^{\geqslant 2}$.

Verra proved that for A general as above, the singular locus of the theta divisor of $\operatorname{Jac}(T_A)$ has dimension 3 (codimension 6). In particular, $\operatorname{Jac}(T_A)$ is not the Jacobian of a curve. Let $\partial\mathscr{A}_g$ be the boundary divisor parametrizing rank-1 degenerations of principally polarized abelian varieties of dimension g. There is a morphism $\rho \colon \partial\mathscr{A}_g \longrightarrow \mathscr{A}_{g-1}$ and the fiber of a point $[(A,\Theta)] \in \mathscr{A}_{g-1}$ is $A/\operatorname{Aut}(A,\Theta)$. One can prove that the image by ρ of the boundary $\partial\mathscr{J}_g := \overline{\mathscr{J}_g} \cap \partial\mathscr{A}_g$ of the locus $\mathscr{J}_g \subset \mathscr{A}_g$ of Jacobians of curves is $\mathscr{J}_{g-1} \subset \mathscr{A}_{g-1}$.

Since the intermediate Jacobian of a general Gushel–Mukai nodal threefold is a $\mathbf{C}^\star$-extension of the intermediate Jacobian of a general Verra threefold, which is not in $\mathscr{J}_{g-1}$, the intermediate Jacobian of a general Gushel–Mukai threefold X is not the Jacobian of a curve: X is therefore irrational.

One can probably get more information about the dimension of the singular locus of the theta divisor of the intermediate Jacobian of a general Gushel–Mukai threefold as follows. Set

$$\mathscr{N}_g^k := \{(A,\Theta) \in \mathscr{A}_g \mid \dim(\operatorname{Sing}(\Theta)) \geqslant k\}.$$

Mumford showed that the boundary $\partial\mathscr{N}_g^k$ is contained in

$$\rho^{-1}(\mathscr{N}_{g-1}^{k-1}) \cup \{((A,\Theta),a) \in \partial\mathscr{A}_g \mid \dim(\mathrm{Sing}(\Theta \cap \Theta_a)) \geqslant k\}. \tag{4.1}$$

Assume $k \geqslant 1$ and that a generates A. If an element $((A,\Theta),a)$ of $\partial\mathscr{N}_g^k$ belongs to the second set in (4.1), it can be shown that $\dim(\mathrm{Sing}(\Theta) \cap \Theta_a) \geqslant k-1$. Therefore, $(A,\Theta) \in \mathscr{N}_{g-1}^{k-1}$.

The intermediate Jacobian (J,Θ) of a very general Verra threefold is simple and belongs to $\mathscr{N}_9^3 \smallsetminus \mathscr{N}_9^4$. The extension class $a \in J$ defined by the intermediate Jacobian of a general nodal Gushel–Mukai threefold is known to be nonzero (one certainly expects it to have infinite order but this does not seem to have been checked).

Proposition 4.1. *If a has infinite order, the singular locus of the theta divisor of the intermediate Jacobian of a general Gushel–Mukai threefold has dimension $\leqslant 4$.*

In this direction, one could also try to prove that the extension class a does not belong to the (known) set of extension classes corresponding to rank-1 degenerations of Prym varieties. This would prove the following conjecture.

Conjecture 4.2. *The intermediate Jacobian of a general Gushel–Mukai threefold X is not a Prym variety. In particular, X is not a conic bundle.*

In this nodal degeneration, the limits of the surfaces $Y_A^{\geqslant 2}$ and $Y_{A^\perp}^{\geqslant 2}$ are special subvarieties (in the sense of Beauville) in the Prym variety $\mathrm{Jac}(T_A)$.

4.2. **The Ferretti degeneration.** Let $S \subset \mathbf{P}(V_4)$ be a smooth quartic surface containing no lines. Ferretti proved that there is a smooth deformation of the surfaces $Y_A^{\geqslant 2}$ to the surface $\mathrm{Bit}(S) \subset \mathsf{Gr}(2,V_4)$ of bitangent lines to S. It goes roughly as follows: consider the degree-6 map

$$\pi_S \colon S^{[2]} \longrightarrow \mathsf{Gr}(2,V_4) \subset \mathbf{P}(\textstyle\bigwedge^2 V_4) = \mathbf{P}^5$$

that sends a point of $S^{[2]}$ to the line that it spans. The pullback of the Plücker polarization is $H-\delta$ (in standard notation); it has Beauville–Bogomolov square 2 and divisibility 1 so, by the irreducibility of the corresponding moduli space of polarized hyper-Kähler fourfolds, it must be a degeneration of double EPW sextics. Actually, it is an actual (degenerate) EPW sextic, which is 3 times the quadric $\mathsf{Gr}(2,V_4) \subset \mathbf{P}^5$. The corresponding Lagrangian is the same for all quartics S: it is given by $A_+ := \mathrm{Sym}^2 V_4 \subset \bigwedge^3(\bigwedge^2 V_4)$.

The canonical involution on the double EPW sextic $S^{[2]}$ is the (Beauville) involution that sends a subscheme of S of length 2 to the residual intersection with S of the line that it spans. Its fixed locus is the surface $\mathrm{Bit}(S)$, which is therefore a degeneration of surfaces $Y_A^{\geqslant 2}$.

Let $W \longrightarrow \mathbf{P}^3$ be the (smooth) double solid branched over S. Welters proved that the variety $F_1(W)$ of lines on W is a connected surface and that the canonical map $F_1(W) \longrightarrow \mathrm{Bit}(S)$ is a double étale cover which is a smooth deformation of the double covers $\widetilde{Y}_A^{\geqslant 2} \longrightarrow Y_A^{\geqslant 2}$ (there is no direct geometric interpretation for that, just the fact that these double covers are associated with the same element of order 2 in $\mathrm{Pic}(Y_A^{\geqslant 2})$).

Welters also showed that the intermediate Jacobian $\mathrm{Jac}(W)$ has dimension 10 and is isomorphic to the Albanese variety of the surface $F_1(W)$. Voisin showed that the singular locus of the theta divisor of $\mathrm{Jac}(W)$ has dimension 5, thereby proving that W is irrational (she also obtained a Torelli theorem for quartic double solids).

The period map for double EPW sextics blows up the (semistable) point $[A_+]$ and maps it onto a Heegner divisor in $\mathscr{D}$ usually denoted as $\mathscr{D}_4$ (O'Grady). It is the only divisor that is invariant by the duality involution.

It is not clear which (degenerate) Gushel–Mukai manifolds should be considered as being associated with the Lagrangian A_+, except perhaps in dimension 5. Consider a smooth quadric $Q \subset \mathbf{P}(V_5)$. The variety $X_Q \subset \mathsf{Gr}(2, V_5)$ of lines in $\mathbf{P}(V_5)$ tangent to Q (the tangential quadratic line complex) is a Gushel–Mukai fivefold, singular along the set of lines contained in Q (that is, along the orthogonal Grassmannian $\mathsf{OGr}(2, V_5)$, which is also the image of the Veronese map $v_2 : \mathbf{P}^3 \hookrightarrow \mathbf{P}^9 \simeq \mathbf{P}(\bigwedge^2 V_5)$). These Gushel–Mukai fivefolds all correspond to the same strictly semistable point in the GIT moduli space $\mathbf{P}(H^0(\mathsf{Gr}(2, V_5), \mathscr{O}_{\mathsf{Gr}(2,V_5)}(2)))/\!\!/ \mathrm{SL}(V_5)$ for ordinary Gushel–Mukai fivefolds and this point should "correspond to" $[A_+]$.

4.3. **Iliev's plan.** Iliev has a grand plan that aims at describing the singularities of the theta divisor of the intermediate Jacobian of a (general) Gushel–Mukai threefold following the method that Voisin used to do the same with quartic double solids. His aim is to prove a Torelli type result which amounts to the (generic) injectivity of the map (2.5). More concretely, the singular locus of a (general) Gushel–Mukai threefold X should more or less be the product of the smooth surfaces $\widetilde{Y}_A^{\geqslant 2}$ and $\widetilde{Y}_{A^\perp}^{\geqslant 2}$. It should be noted that it is very difficult to describe the singularities of a theta divisor when the ambient variety is not a Prym variety: to the best of our knowledge, the only case where it has been done was in Voisin's proof of the Torelli theorem for quartic double solids.

Iliev's plan goes roughly as follows (almost everything that follows is conjectural).

Let X be a (general) Gushel–Mukai threefold. Recall that the surface $\widetilde{Y}_A^{\geqslant 2}$ is (the minimal model of) the surface $C_2^0(X)$ that parametrizes conics contained in X. Let $C_3^0(X)$ be the 3-dimensional Hilbert scheme of twisted cubics contained in X and let $C_5^1(X)$ be the 5-dimensional Hilbert scheme of elliptic quintics contained in X, with their Abel–Jacobi maps

$$\mathrm{AJ}_2\colon C_2^0(X) \longrightarrow \mathrm{Jac}(X) \quad , \quad \mathrm{AJ}_3\colon C_3^0(X) \longrightarrow \mathrm{Jac}(X) \quad , \quad \mathrm{AJ}_5\colon C_5^1(X) \longrightarrow \mathrm{Jac}(X),$$

only defined up to translation.

Conjecture 4.3 (Iliev). *Let X be a general Gushel–Mukai threefold.*

(a) *There exists a theta divisor* $\Theta \subset \mathrm{Jac}(X)$ *such that* $\mathrm{AJ}_2(C_2^0(X)) + \mathrm{AJ}_3(C_3^0(X)) \subset \Theta$.
(b) *The image* $\mathrm{AJ}_5(C_5^1(X))$ *is a surface* Z_X *contained in* $\mathrm{Sing}(\Theta)$.[4]
(c) *This surface* Z_X *is isomorphic to* $\mathrm{AJ}_2(C_2^0(X)) \simeq \widetilde{Y}_A^2$.

[4] Note that a conic and a cubic (both contained in X) that meet in two points define a point of $C_5^1(X)$.

(d) *When X describes the fiber $Y^2_{A^\perp}$ of the map p_3, the union Z_A of all these surfaces Z_X is a 4-dimensional component of* $\mathrm{Sing}(\Theta)$ *with a morphism* $Z_A \twoheadrightarrow \widetilde{Y}^2_{A^\perp}$ *with fibers all isomorphic to* $\widetilde{Y}^2_A$.

(e) *The component Z_A is the unique 4-dimensional component of* $\mathrm{Sing}(\Theta)$.

As in Voisin's proof, the first step would be to study the sections of the pullbacks by AJ_2 of the various translates of the theta divisor: for $a \in \mathrm{Jac}(X)$ general, one has $h^0(C^0_2(X), \mathrm{AJ}_2^* \Theta_a) = 1$. One then establishes a sort of converse to that statement: the translates Θ_a that contain the surface $\mathrm{AJ}_2(C^0_2(X))$ are "more or less" those for which $h^0(C^0_2(X), \mathrm{AJ}_2^* \Theta_a) > 1$. One uses this criterion to construct a family of translates of $\mathrm{AJ}_2(C^0_2(X))$ contained in Θ (this is item (a) above).

Next, $H^0(\Theta_a, \Theta_a|_{\Theta_a})$ is a 10-dimensional vector space that defines the Gauss map of Θ_a and its base locus is $\mathrm{Sing}(\Theta_a)$ (it is given by the partial derivatives of an equation of Θ_a). The next step is to study, for translates Θ_a that contain the surface $\mathrm{AJ}_2(C^0_2(X))$, the image of the restriction

$$H^0(\Theta_a, \Theta_a|_{\Theta_a}) \longrightarrow H^0(\Theta_a, \Theta_a|_{\mathrm{AJ}_2(C^0_2(X))})$$

The base loci of the image corresponds to singular points of Θ_a contained in $\mathrm{AJ}_2(C^0_2(X))$ or, equivalently, to singular points of Θ contained in the translate $\mathrm{AJ}_2(C^0_2(X)) - a$. This points to a way of proving item (b) above.

Unfortunately, it seems already very hard to prove even what we just stated.

5. Derived categories of Gushel–Mukai varieties

In this section, we study semiorthogonal decompositions for Gushel–Mukai varieties. We define the *Gushel–Mukai category* of a Gushel–Mukai variety X as a particular subcategory of its derived category. We discuss some properties of Gushel–Mukai categories, culminating with a duality theorem that relates Gushel–Mukai categories of some pairs of Gushel–Mukai varieties.

5.1. Introduction to Lefschetz decompositions.

Since the main subject of this section is a particular semiorthogonal decomposition of the derived category of coherent sheaves on a Gushel–Mukai variety, we need to introduce various notions. First, we fix notation for the remainder of the section:

- for us, all categories are assumed to be triangulated, all functors are assumed to be derived, and the base field is $\mathbf{C}$;
- $\mathrm{D^b}(X)$ is the bounded derived category of coherent sheaves on X;
- a subcategory $\mathscr{A} \subset \mathrm{D^b}(X)$ is called *admissible* if the embedding functor $i\colon \mathscr{A} \hookrightarrow \mathrm{D^b}(X)$ has both left and right adjoint functors.

Definition 5.1. A sequence $\mathscr{A}_1, \dots, \mathscr{A}_n$ of subcategories is a *semiorthogonal decomposition* (abbreviated to SOD) of X if:

- each $\mathscr{A}_i \subset \mathrm{D^b}(X)$ is admissible;
- $\mathrm{Hom}(A_j, A_i) = 0$ for any $A_j \in \mathscr{A}_j$, $A_i \in \mathscr{A}_i$ if $j > i$;
- $\{\mathscr{A}_1, \dots, \mathscr{A}_n\}$ generates $\mathrm{D^b}(X)$.

We write $\mathrm{D^b}(X) = \langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle$.

Definition 5.2. Let $f \colon X \longrightarrow \mathbf{P}(V)$ be a morphism. A *Lefschetz decomposition* of $\mathrm{D^b}(X)$ is a semiorthogonal decomposition of $\mathrm{D^b}(X)$ of the form

$$\mathrm{D^b}(X) = \langle \mathscr{A}_0, \mathscr{A}_1(1), \dots, \mathscr{A}_{m-1}(m-1) \rangle,$$

where $\mathscr{A}_0 \supset \mathscr{A}_1 \supset \cdots \supset \mathscr{A}_{m-1}$ is a chain of admissible subcategories and the twist (i) is tensor multiplication by $f^*\mathscr{O}_{\mathbf{P}(V)}(i)$.

Fact 5.3. *A Lefschetz decomposition is uniquely determined by the first subcategory $\mathscr{A}_0$.*

Definition 5.4. A Lefschetz decomposition is *rectangular* if $\mathscr{A}_0 = \cdots = \mathscr{A}_{m-1}$.

Example 5.5. The decomposition $\mathrm{D^b}(\mathbf{P}^n) = \langle \mathscr{O}, \mathscr{O}(1), \dots, \mathscr{O}(n) \rangle$ is a rectangular Lefschetz decomposition with $\mathscr{A}_0 = \cdots = \mathscr{A}_n = \langle \mathscr{O} \rangle$.

Example 5.6. Let $Q \subset \mathbf{P}^{n+1}$ be a smooth quadric. For a particular subcategory $\mathscr{A}_Q \subset \mathrm{D^b}(Q)$, which is generated by one or two spinor bundles depending on the parity of n, there exists a decomposition

$$\mathrm{D^b}(Q) = \langle \mathscr{A}_Q, \mathscr{O}(1), \dots, \mathscr{O}(n) \rangle.$$

We can consider this as a Lefschetz decomposition with

$$\mathscr{A}_0 = \langle \mathscr{A}_Q, \mathscr{O}(1) \rangle \quad , \quad \mathscr{A}_1 = \cdots = \mathscr{A}_{n-1} = \langle \mathscr{O}(1) \rangle.$$

This decomposition is evidently not rectangular, but only because the first component $\mathscr{A}_0$ is larger than $\mathscr{A}_1$ by $\mathscr{A}_Q$. We will see many decompositions like that in the rest of these notes.

5.2. **Reminder on Gushel–Mukai varieties.** Recall from Definition 1.2 that an ordinary Gushel–Mukai variety is a smooth complete intersection

$$X = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W) \cap Q$$

of subvarieties of $\mathbf{P}(\bigwedge^2 V_5)$, where W is a linear subspace in $\bigwedge^2 V_5$ and Q is a quadric in $\mathbf{P}(W)$.

Such a variety X can be alternatively described in terms of its *Lagrangian data set*: a triple (V_6, V_5, A), where

- $V_5 \subset V_6$ is a hyperplane;
- $A \subset \bigwedge^3 V_6$ is a Lagrangian subspace.

A quick reminder on the connection between those two descriptions: suppose we are given X, W, Q. The space $H^0(\mathbf{P}(W), \mathscr{I}_X(2))$ of quadrics in $\mathbf{P}(W)$ containing X is V_6. The condition that a quadric contain the larger subvariety $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W)$ defines a hyperplane $V_5 \subset V_6$.

The quadric $Q \subset \mathbf{P}(W)$, as an element of V_6, is by definition in the complement $V_6 \smallsetminus V_5$. So we get a direct sum decomposition $V_6 = V_5 \oplus \mathbf{C}Q$ as in (1.1), which induces a decomposition

$$\bigwedge^3 V_6 = \bigwedge^3 V_5 \oplus (\bigwedge^2 V_5 \otimes Q).$$

In terms of Lagrangian data, the space on the left, $\bigwedge^3 V_6$, contains the subspace A. On the right side, the summand $\bigwedge^2 V_5 \otimes Q$ has a subspace $W \otimes Q$. Those two subspaces are related as in (1.2).

5.3. **Semiorthogonal decomposition for Gushel–Mukai manifolds.** A lot is known about semiorthogonal decompositions of Grassmannians and their linear sections. We will need the following statement.

Theorem 5.7. *Let $\mathscr{U}$ be the tautological subbundle on $\mathsf{Gr}(2, V_5)$. Denote by $\mathscr{B}$ the subcategory $\langle \mathscr{O}, \mathscr{U}^\vee \rangle \subset \mathrm{D}^{\mathrm{b}}(\mathsf{Gr}(2, V_5))$. Then,*

- *$\mathrm{D}^{\mathrm{b}}(\mathsf{Gr}(2,5)) = \langle \mathscr{B}, \dots, \mathscr{B}(4) \rangle$ is a rectangular Lefschetz decomposition;*
- *if $i\colon M \hookrightarrow \mathsf{Gr}(2,5)$ is a smooth linear section of dimension $N \geqslant 3$, then i^* induces an embedding of $\mathscr{B} \subset \mathrm{D}^{\mathrm{b}}(\mathsf{Gr}(2,5))$ into $\mathrm{D}^{\mathrm{b}}(M)$, and $\mathrm{D}^{\mathrm{b}}(M) = \langle \mathscr{B}, \dots, \mathscr{B}(N-2) \rangle$ is a rectangular Lefschetz decomposition.*

Passing to a quadric section of $\mathsf{Gr}(2,5) \cap \mathbf{P}(W)$ complicates things. Instead of a rectangular Lefschetz decomposition, we get a decomposition where the first component is more complicated than the others (compare with Example 5.6).

Theorem 5.8. *Let X be an ordinary Gushel–Mukai manifold of dimension $n \geqslant 3$ and let $\gamma\colon X \longrightarrow \mathsf{Gr}(2, V_5)$ be its Gushel map. Then γ^* induces an embedding of the subcategory $\mathscr{B} \subset \mathrm{D}^{\mathrm{b}}(\mathsf{Gr}(2, V_5))$ into $\mathrm{D}^{\mathrm{b}}(X)$, and there is a semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathscr{A}_X, \mathscr{B}, \dots, \mathscr{B}(n-3) \rangle,$$

where $\mathscr{A}_X$ is the right orthogonal of $\langle \mathscr{B}, \dots, \mathscr{B}(n-3) \rangle$.

Roughly speaking, this result says that the most interesting part of the derived category $\mathrm{D}^{\mathrm{b}}(X)$ is the subcategory $\mathscr{A}_X$, with the rest essentially "induced" from $\mathrm{D}^{\mathrm{b}}(\mathsf{Gr}(2, V_5))$.

Definition 5.9. The category $\mathscr{A}_X$ is called the *Gushel–Mukai category* of X.

It turns out that for even-dimensional Gushel–Mukai manifolds, the Gushel–Mukai category behaves in many ways like the derived category of a K3 surface. In particular, it has a "Serre duality" functor that is the shift by 2, like in the K3 surface case.

Definition 5.10. A *Serre functor* of a (triangulated) category $\mathscr{D}$ is an endofunctor $S\colon \mathscr{D} \longrightarrow \mathscr{D}$ with a natural isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(F, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(G, S(F))^\vee$$

for each pair of objects $F, G \in \mathscr{D}$.

A Serre functor is uniquely determined by this property, if it exists.

Example 5.11. For any smooth proper variety X, the category $\mathrm{D^b}(X)$ has a Serre functor given by Serre duality:

$$S_{\mathrm{D^b}(X)}(F) = F \otimes \omega_X[\dim(X)].$$

In particular, for the derived category of a K3 surface, the Serre functor is just the shift [2].

On each Gushel–Mukai category, the Serre functor exists and is given as follows.

Theorem 5.12 ([KP1])**.** *Let X be a Gushel–Mukai manifold of dimension n.*

(1) *If n is even, $S_{\mathscr{A}_X} \simeq [2]$ ("noncommutative K3 surface").*
(2) *If n is odd, $S_{\mathscr{A}_X} \simeq \sigma \circ [2]$ for some nontrivial involution σ of $\mathscr{A}_X$. If X is a special Gushel–Mukai manifold, σ is induced from the double cover involution on X.*

5.4. **Duality of Gushel–Mukai categories.** The description of Gushel–Mukai manifolds in terms of Lagrangian data sets highlights some relations between different Gushel–Mukai manifolds.

Definition 5.13. If X and X' are Gushel–Mukai manifolds such that for their Lagrangian data sets, there exist an isomorphism $\varphi\colon V_6 \xrightarrow{\sim} (V_6')^\vee$ with $\varphi(A) = (A')^\perp$, then X and X' are called

- *dual* if $\dim(X) = \dim(X')$;
- *generalized dual* if $\dim(X) \equiv \dim(X') \pmod 2$.

Duality between Gushel–Mukai manifolds has strong geometric implications.

Theorem 5.14 ([DK5])**.** *If X and X' are dual Gushel–Mukai manifolds of dimension $\geqslant 3$, they are birationally isomorphic.*

Similarly, generalized duality implies a relation between Gushel–Mukai categories. This categorical generalization was obtained by Kuznetsov and Perry.

Theorem 5.15 ([KP2])**.** *If X and X' are generalized dual Gushel–Mukai manifolds, there is an equivalence of categories $\mathscr{A}_X \xrightarrow{\sim} \mathscr{A}_{X'}$.*

This result is especially interesting because it concerns manifolds of possibly different dimensions. Studying derived categories of higher-dimensional manifolds is usually difficult, so any kind of a dimensional reduction can be useful.

In the remainder of the section, we will discuss the duality result Theorem 5.15 and sketch a proof for some particular pairs of dual Gushel–Mukai manifolds.

Remark 5.16. The set of isomorphism classes of generalized duals to X is the quotient $\mathbf{P}(V_6(X))/\mathrm{PGL}(V_6)_A$, with $q \in \mathbf{P}(V_6(X))$ corresponding to a generalized dual of dimension $5 - \dim(\ker(q))$. From linear algebra considerations about possible dimensions of kernels, we see that the duality from Theorem 5.15 predicts that

- if $\dim(X) = 6$, the category $\mathscr{A}_X$ is equivalent to the Gushel–Mukai category of a Gushel–Mukai fourfold;
- if $\dim(X) = 5$, the category $\mathscr{A}_X$ is equivalent to the Gushel–Mukai category of a Gushel–Mukai threefold;
- if $\dim(X) = 4$ and there exists $q \in V_6$ with $\dim(\ker(q)) = 3$, the category $\mathscr{A}_X$ is equivalent to the derived category of a K3 surface.

As a kind of a mix between Theorems 5.14 and 5.15, there is the following conjecture.

Conjecture 5.17. *If X is a rational Gushel–Mukai fourfold, the category $\mathscr{A}_X$ is equivalent to the derived category of a K3 surface.*

5.5. **Proof of duality for some Gushel–Mukai fourfolds.** The proof of Theorem 5.15 relies on the theory of homological projective duality (often abbreviated as HPD). This theory is being actively developed and the general proof in [KP2] involves a powerful but quite abstract machinery. However, for a certain class of Gushel–Mukai fourfolds, the duality statement was proved earlier in [KP1] using a less advanced version of HPD but relying more on the geometry of Gushel–Mukai manifolds. In this section, we will give a rough sketch of that argument. We start with a brief introduction to homological projective duality.

5.5.1. *The role of homological projective duality.* We begin with a couple of words about homological projective duality. This theory, introduced in [K1] and later developed much further, is quite technical. A nice introduction can be found in [K2]. Instead of a rigorous formulation, we will give an imprecise version with lots of hand-waving. To apply the theory of HPD, we should know the following three pieces of information:

(1) what does it mean for two varieties to be homologically projectively dual?
(2) which examples are known?
(3) most importantly, what does this condition imply about derived categories?

We will say something about those questions below.

Let $X \subset \mathbf{P}(V)$ and $Y \subset \mathbf{P}(V^\vee)$ be varieties, both equipped with Lefschetz semiorthogonal decompositions. There is a definition of what it means for X and Y (with the specified semiorthogonal decompositions) to be *homologically projectively dual* to each other. Roughly speaking, it is a certain relation between the derived category of Y and the derived category of the universal hyperplane section of X. This condition happens to be symmetric in X and Y. We will not need the precise definition, since we only care about examples.

Specifically, we need only one example: homological projective duality for the Grassmannian $\mathsf{Gr}(2, 5)$.

Theorem 5.18. *The Plücker embeddings $\mathsf{Gr}(2, V_5) \subset \mathbf{P}(\bigwedge^2 V_5)$ and $\mathsf{Gr}(2, V_5^\vee) \subset \mathbf{P}(\bigwedge^2 V_5^\vee)$ are homologically projectively dual.*

The most important consequence of the homological projective duality between $X \subset \mathbf{P}(V)$ and $Y \subset \mathbf{P}(V^\vee)$ is that for any linear subspace $L \subset V$, the derived categories of the linear sections $X \cap L$ and $Y \cap L^\perp$ have induced semiorthogonal decompositions with a common nontrivial component, where "nontrivial" means that it does not come from $\mathrm{D^b}(X)$ nor $\mathrm{D^b}(Y)$, in a certain precise sense. Well, that is not quite true: the intersections $X \cap L$ and $Y \cap L^\perp$ should both be of the expected dimensions, and ideally also smooth, but we omit most technical conditions in this crude introduction. This even works in families: instead of a single linear section, we may consider the incidence variety for a family of linear sections, with the same conclusion.

Theorem 5.19 (Main theorem of HPD)**.** *Let $X \subset \mathbf{P}(V)$ and $Y \subset \mathbf{P}(V^\vee)$ be homologically projectively dual varieties. Let $\mathscr{F}$ be a family of linear subspaces of constant dimension in $\mathbf{P}(V)$ parametrized by a smooth and proper variety B. Consider the family $\mathscr{F}^\perp$ of their orthogonal complements considered as linear subspaces in $\mathbf{P}(V^\vee)$, parametrized by the same base B. Consider the incidence varieties*

$$X_{\mathscr{F}} := \{(x,b) \mid x \in X, b \in B, x \in \mathscr{F}_b\} \subset X \times B,$$
$$Y_{\mathscr{F}^\perp} := \{(y,b) \mid y \in Y, b \in B, y \in \mathscr{F}_b^\perp\} \subset Y \times B.$$

Then there exist induced semiorthogonal decompositions of $\mathrm{D^b}(X_{\mathscr{F}})$ and $\mathrm{D^b}(Y_{\mathscr{F}^\perp})$ with components of two types:

(1) *pieces "induced" from $\mathrm{D^b}(B)$ and the Lefschetz decompositions of X and Y;*
(2) *a nontrivial component $\mathscr{A}$ that is the same for both $\mathrm{D^b}(X_{\mathscr{F}})$ and $\mathrm{D^b}(Y_{\mathscr{F}^\perp})$.*

Remark 5.20. Warning: we omit many necessary conditions on, for example, singularities of linear sections of X, expected dimensions for the intersections, etc. Besides, the words "pieces induced from" in this statement have no mathematical meaning: the only sensible way to state the result precisely is to give an explicit description of the semiorthogonal decompositions, which we avoid.

In Theorem 5.15, we want to show that two different derived categories have a common component so, at least superficially, the main theorem of HPD seems to be useful in our situation! Unfortunately, it is not directly applicable since Gushel–Mukai manifolds are not (families of) linear sections of Grassmannians.

Remark 5.21. If $L \subset V$ is a linear subspace, $\mathbf{P}(L) \subset \mathbf{P}(V)$ is HPD to $\mathbf{P}(L^\perp) \subset \mathbf{P}(V^\vee)$. Theorem 5.19 above is a relation between the derived categories of $X \cap \mathbf{P}(L)$ and $Y \cap \mathbf{P}(L^\perp)$. In [KP2], a much more general statement was proved: if $X, X' \subset \mathbf{P}(V)$ are HPD, respectively, to $Y, Y' \subset \mathbf{P}(V^\vee)$ then, under some assumptions, the derived categories $\mathrm{D^b}(X \cap X')$ and $\mathrm{D^b}(Y \cap Y')$ also have a common nontrivial component. Using this generalization, Kuznetsov and Perry proved Theorem 5.15 using homological projective duality for $\mathsf{Gr}(2,5)$ (Theorem 5.18), for linear subspaces, and for quadrics, since Gushel–Mukai manifolds arise as intersections of those three. We will follow a different strategy.

5.5.2. *Dual (generalized) Gushel–Mukai manifolds, geometrically.* We recall the construction of (generalized) dual Gushel–Mukai manifolds. By Definition 5.13, starting from a Lagrangian data set (V_6, V_5, A), we want to construct a triple (V_6', V_5', A') such that

$$V_6' \simeq V_6^\vee \quad , \quad A' = A^\perp.$$

We only need to choose a hyperplane V_5' in $V_6^\vee$ to complete the picture. This is the same as a point in $\mathbf{P}(V_6)$, that is, a quadric in $\mathbf{P}(W)$ containing the Gushel–Mukai manifold X. Thus, for any $q \in \mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$, we define a dual Gushel–Mukai manifold $X_q^\vee$ as the one corresponding to the Lagrangian data set $(V_6^\vee, q^\perp, A^\perp)$.

We can now state the theorem proved in [KP1].

Theorem 5.22 ([KP1])**.** *Let X be an ordinary Gushel–Mukai fourfold. For any $q \in \mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$ such that $\dim(\ker(q)) = 3$, if $X_q^\vee$ is a smooth K3 surface, there exists an equivalence of categories $\mathscr{A}_X \simeq \mathrm{D^b}(X_q^\vee)$.*

Remark 5.23. This is a special case of the duality of Theorem 5.15.

Remark 5.24. Not every Gushel–Mukai fourfold has a q as in the theorem. In fact, the set of Gushel–Mukai fourfolds for which such a q exists is a divisor in the moduli space: in the notation of Definition 2.1, such a q corresponds to a point of Y_A^3.

An important geometric observation is that if we consider $q \in \mathbf{P}(V_6)$ as in the theorem as a quadric in $\mathbf{P}(W)$ containing $X \subset \mathbf{P}(W)$ then, since $q \notin \mathbf{P}(V_5)$, we have $X = \mathsf{Gr}(2,5) \cap \mathbf{P}(W) \cap \{q = 0\}$. Building upon this observation, Debarre and Kuznetsov proved the following, even more explicit description.

Theorem 5.25 ([DK5])**.** *Let X and q be as in Theorem 5.22. There exists an isomorphism $V_5(X) \simeq V_5(X_q^\vee)^\vee$ under which the embedding $X_q^\vee \hookrightarrow \mathbf{P}(\bigwedge^2 V_5(X_q^\vee))$ is identified with the intersection*

$$X_q^\vee = \mathsf{Gr}(2, V_5^\vee) \cap Q^\vee,$$

where $Q^\vee \subset \mathbf{P}(\bigwedge^2 V_5^\vee)$ is the projective dual to the subvariety $\{q = 0\} \subset \mathbf{P}(\bigwedge^2 V_5)$.

The last line needs clarification. Generally, the projective dual to a smooth subvariety of a projective space is the closure of the set of tangent hyperplanes to it, as a subvariety in the dual projective space. For example, for a smooth quadric hypersurface in some $\mathbf{P}(V)$, the projective dual variety in $\mathbf{P}(V^\vee)$ is exactly the dual smooth quadric, that is, the one corresponding to the nondegenerate pairing on $V^\vee$ induced by the pairing on V. In Theorem 5.22, instead of a smooth quadric hypersurface, we have a singular quadric defined only on a subspace $W \subset \bigwedge^2 V_5$. We will just use the following description as a definition.

Definition 5.26. Let U be a vector space and let $K \subset W \subset U$ be subspaces. Let q be a quadratic form on W such that $K = \ker(q)$. Denote by $Q \subset \mathbf{P}(U)$ the image of the quadric $\{q = 0\} \subset \mathbf{P}(W)$ under the linear embedding. Then Q is a cone in $\mathbf{P}(W)$ over a smooth quadric $\bar{Q}$ in the quotient space W/K, with vertex $\mathbf{P}(K)$. Consider the dual chain

$$0 \subset W^{\perp} \subset K^{\perp} \subset U^{\vee}$$

of vector spaces. Let $\bar{Q}^{\vee}$ be the smooth quadric in the quotient $K^{\perp}/W^{\perp} \simeq (W/K)^{\vee}$ dual to $\bar{Q}$. Let $Q^{\vee}$ be the cone in $\mathbf{P}(K^{\perp})$ over the quadric $\bar{Q}^{\vee}$ in the quotient space $K^{\perp}/W^{\perp}$, with vertex $\mathbf{P}(W^{\perp})$. Then $Q \subset \mathbf{P}(U)$ and $Q^{\vee} \subset \mathbf{P}(U^{\vee})$ are projectively dual subvarieties.

Remark 5.27. The proof of this description is basically an exercise in linear algebra. For example, here is why the dual subvariety should lie in $\mathbf{P}(K^{\perp}) \subset \mathbf{P}(U^{\vee})$. Consider a hyperplane $H \subset \mathbf{P}(U)$ that passes through some smooth point $p \in Q$ and contains the tangent space to Q at p. Since Q is a cone with vertex $\mathbf{P}(K) \subset \mathbf{P}(U)$, we see that H necessarily contains the subspace $\mathbf{P}(K)$. Thus H as a point in $\mathbf{P}(U^{\vee})$ lies in the subspace $\mathbf{P}(K^{\perp})$.

5.5.3. *Combining HPD and geometry.* In Theorem 5.22, we start with an ordinary Gushel–Mukai fourfold X. It is an intersection $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W) \cap Q$ for some (singular) quadric $Q \subset \mathbf{P}(W)$. The key idea in the proof is to associate with the quadric Q the family of maximal isotropic linear subspaces on it. Then, instead of X, we consider the incidence variety

$$\widetilde{X} = \{(x, I) \mid x \in X,\ I \subset Q \subset \mathbf{P}(W) \ \text{ maximal isotropic subspace}\}.$$

More precisely, maximal isotropic subspaces on Q form two families, both parametrized by $\mathbf{P}^3$, and we pick only subspaces from one of the families.

The variety $\widetilde{X}$ comes with two projections. The first one, $\pi_X \colon \widetilde{X} \longrightarrow X$, forgets the subspace I. Each point on X lies on a $\mathbf{P}^1$-family of maximal isotropic subspaces on Q. Thus the morphism π_X is a $\mathbf{P}^1$-fibration. The second map, $p_X \colon \widetilde{X} \longrightarrow \mathbf{P}^3$, where $\mathbf{P}^3$ is the base of the chosen family of maximal isotropic subspaces, forgets the point x. The fiber over a point $I \in \mathbf{P}^3$ is exactly the intersection $X \cap I$, but since $I \subset Q$, this may be directly described as $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W) \cap I$, so this is a linear section of the Grassmannian $\mathsf{Gr}(2, 5)$.

Denote by Y the dual variety $X_q^{\vee} = \mathsf{Gr}(2, V_5^{\vee}) \cap Q^{\vee}$ from Theorem 5.25. Using the explicit description of $Q^{\vee}$ in Definition 5.26, we see that $Q^{\vee}$ also has two families of maximal isotropic subspaces, and, moreover, if $I \subset \bigwedge^2 V_5$ is a maximal isotropic subspace on Q, then $I^{\perp} \subset \bigwedge^2 V_5^{\vee}$ is a maximal isotropic subspace on $Q^{\vee}$.

Thus we can construct $\widetilde{Y}$ analogously to $\widetilde{X}$, as an incidence variety of maximal isotropic subspaces on $Q^{\vee}$, and we have maps

$$\mathbf{P}(\textstyle\bigwedge^2 V_5) \times \mathbf{P}^3 \supset \widetilde{X} \xrightarrow{p_X} \mathbf{P}^3 \xleftarrow{p_Y} \widetilde{Y} \subset \mathbf{P}(\textstyle\bigwedge^2 V_5^{\vee}) \times \mathbf{P}^3.$$

As mentioned above, both p_X and p_Y are families of linear sections of the Grassmannian $\mathsf{Gr}(2, 5)$; moreover, when considered as families of linear subspaces of $\mathbf{P}(\bigwedge^2 V_5)$ and $\mathbf{P}(\bigwedge^2 V_5^{\vee})$, they are orthogonal subspaces. This is exactly the situation where we can apply the main theorem of HPD (Theorem 5.19)!

So we finally get relations between the derived categories $\mathrm{D^b}(\widetilde{X})$ and $\mathrm{D^b}(\widetilde{Y})$. This is not quite what we wanted, but now we can gradually improve the things we know.

- From Theorem 5.19, we know that $\mathrm{D^b}(\widetilde{X})$ and $\mathrm{D^b}(\widetilde{Y})$ have a common interesting subcategory; let us call it $\widetilde{\mathscr{A}}$.
- Since $\widetilde{X}$ is a $\mathbf{P}^1$-family over X, the pullback $\pi_X^*\colon \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(\widetilde{X})$ is an embedding of categories. In particular, if $\mathscr{A}_X \subset \mathrm{D^b}(X)$ is the Gushel–Mukai category of X, we can consider $\pi_X^*(\mathscr{A}_X)$ as a subcategory of $\mathrm{D^b}(\widetilde{X})$.
- Similarly, $\mathrm{D^b}(Y)$ embeds into $\mathrm{D^b}(\widetilde{Y})$ since $\pi_Y\colon \widetilde{Y} \longrightarrow Y$ is also a $\mathbf{P}^1$-family.
- Thus, in $\mathrm{D^b}(\widetilde{X})$, we have two subcategories: $\widetilde{\mathscr{A}}$ from the main theorem of HPD, and $\pi_X^*(\mathscr{A}_X)$ arising from a Gushel–Mukai category of X. In fact, both of those subcategories are components in some (explicit) semiorthogonal decompositions. After performing a complicated sequence of mutations (certain procedures that transform a semiorthogonal decomposition into another one) of those two semiorthogonal decompositions of $\mathrm{D^b}(\widetilde{X})$, one can deduce that the subcategories $\widetilde{\mathscr{A}}$ and $\pi_X^*(\mathscr{A}_X)$ are equivalent by "lining them up" inside $\mathrm{D^b}(\widetilde{X})$.
- Similarly, in $\mathrm{D^b}(\widetilde{Y})$, we have two subcategories: $\widetilde{\mathscr{A}}$ from the main theorem of HPD, and a copy of $\mathrm{D^b}(Y)$ pulled back along π_Y. Similarly, after a sequence of mutations of semiorthogonal decompositions of $\mathrm{D^b}(\widetilde{Y})$, one can deduce that $\widetilde{\mathscr{A}}$ is equivalent to $\mathrm{D^b}(Y)$.
- Thus eventually we get $\mathscr{A}_X \simeq \pi_X^*(\mathscr{A}_X) \simeq \widetilde{\mathscr{A}} \simeq \mathrm{D^b}(Y)$, as claimed in Theorem 5.22. QED!

The most complicated part of this procedure is figuring out a way to compare two subcategories, $\widetilde{A}$ and $\pi_X^*(\mathscr{A}_X)$, of $\mathrm{D^b}(\widetilde{X})$. This requires a large amount of computations (of Hom between objects in derived categories).

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